the hypersonic approximation $2H \approx u^2$ at the edge of the viscous layer in conjunction with $\mu \sim T^s$ with $s < 1$. If the shock wave is treated as a discontinuity, the proper boundary conditions require that the shear stress at the shock wave correspond to the vorticity due to shock curvature. This condition cannot, in general, be imposed on a solution based on the boundary layer equations and is not met here in the region closest to the leading edge ($\xi \rightarrow 0$) where, moreover, the no-slip condition is invalid also. It may be noted that $(\mu\ \partial u/\partial y)_s \sim p(x)\ (\partial^2 F/\partial Y^2)_s/(x)^{1/2}\eta_s(x)$ decreases rapidly with increasing ξ and falls to zero for $\xi \approx 4$ where, however, the convergence of the series for $(\partial^2 F/\partial Y^2)_s$ is rather poor. An alternative treatment of the leading edge problem is given by Street,[3] who proposes to satisfy boundary conditions at infinity rather than at the shock wave, which appears to imply that his flow model is not one in which the viscous layer and the shock layer coincide.

Slip and temperature jump are negligible if $\xi \gg (\gamma + 1)s/\gamma$. For a cold wall in the hypersonic limit ($s \rightarrow 0$; $t = 1$), convergence of the series for pressure and velocity requires that $\xi < 10$ approximately. The solution given here is therefore valid in $h_w/H_\infty \ll Re/M^2 < 10\gamma/(\gamma + 1)$ or for $\chi \approx O(M^2)$.

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Optimization of Stochastic Trajectories

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Nomenclature

a_1, a_2, a_3 = weighting factors in payoff function
b = thrust per unit initial mass
$\mathbf{f}(\ )$ = vector function of ( )
$\mathbf{F}$ = matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{y}, F_{ij} = \partial f_i/\partial y_j$
g = gravitational acceleration
$\mathbf{G}$ = matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{z}, G_{il} = \partial f_i/\partial z_l$
$\mathbf{H}$ = measurement inference matrix; converts state vector to expected measurement vector
$\mathbf{K}$ = gain matrix for estimator
m = mass
$\mathbf{P}$ = covariance matrix of uncertainty in state vector
$\mathbf{Q}$ = covariance matrix of uncertainty in control vector
$\mathbf{R}$ = covariance matrix of uncertainty in measurement vector
t = time
u = horizontal velocity
v = vertical velocity
$\mathbf{X}$ = measurement vector
y = altitude
$\mathbf{y}$ = state vector $\{y,v,u\}$
$\mathbf{Y}$ = uncertainty in state vector
$\mathbf{z}$ = control vector $\{b,\theta\}$
$\mathbf{Z}$ = uncertainty in control vector
γ = fractional mass flow per second
δ = perturbation
Δ = increment
$\boldsymbol{\lambda}$ = matrix of adjoint variables
Λ = transition matrix for $\boldsymbol{\lambda}$
σ = standard deviation
ϕ = payoff quantity
θ = thrust angle, measured from horizontal

Subscripts

d = final descent phase
0 = initial (ignition) time
1 = final (end of main phase) time

Superscripts

$\dot{}$ = time derivative
$'$ = matrix transpose
-1 = matrix inversion

Introduction

In recent years, much study has gone into the problems of trajectory optimization[1–3] and the problem of optimal control around a reference trajectory.[4] The control analyses are linearized; hence their application to extremal trajectories gives indeterminate results. This has led to the formulation of second-order optimal control theory.[5] These analyses are based on deterministic situations; that is, the state of the system is known, and control changes can be applied exactly. Real problems, however, are not deterministic. The state only can be inferred from a possibly incomplete set of noisy measurements, and the control variables themselves are subject to random variation. This fact has led to the formulation of optimum linear filters.[6]§

Unfortunately, this acceptance of the true, stochastic nature of the problem always has come after the reference trajectory has been chosen. For some operational criteria this is correct, but if the criterion for choice is the optimality of the trajectory, and if the performance is affected by the statistics of the situation, the effect of the statistics should be incorporated into the optimization procedure. The problem cannot be discussed in generality but must be illustrated through a specific example.

As an example, the authors have examined the problem of achieving a soft landing on the moon, which was considered deterministically in Ref. 7. Such a trajectory very likely will consist of two phases: first a main phase during which most of the energy of the vehicle is dissipated, followed by a terminal phase in which the vehicle descends slowly to touchdown. This second phase is less efficient at energy dissipation than the main phase; hence it should begin at as low an energy level as is feasible. However, the transition point must be tied to the uncertainty in the altitude and velocity of the vehicle. Thus it is apparent that a main-phase trajectory that reduces uncertainties can reduce terminal phase propellant consumption and, in fact, can reduce the total expected propellant consumption if the main phase consumption is not affected seriously. A more detailed presentation of this work is given in Ref. 8.

Equations of Motion and the Filter

The vehicle is assumed to be a constant thrust rocket moving in a uniform, parallel gravity field, with no other external

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§ In Ref. 6, Kalman has shown that the optimal linear control and the optimal linear filter are duals.

forces. The equations of motion are

$$\dot{y} = v$$
$$\dot{v} = [b \sin\theta/(1 - \gamma t)] - g$$
$$\dot{u} = b \cos\theta/(1 - \gamma t)$$

or, in matrix form,

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y},\mathbf{z},t) \quad (1)$$

and the linearized perturbation equations are

$$\delta\dot{\mathbf{y}} = \mathbf{F}\delta\mathbf{Y} + \mathbf{G}\delta\mathbf{Z}$$

where boldface characters are used to indicate matrices, including vectors. The trajectory uncertainty is estimated by a model of the perturbation equations with a correction due to the difference between the value of $\delta\mathbf{y}$ predicted from previous information and the value inferred from current measurements:

$$\dot{\mathbf{Y}} = \mathbf{FY} + \mathbf{GZ} + \mathbf{K}[\mathbf{X} - \mathbf{HY}]$$

Such a system is shown in Fig. 1. According to Kalman,[6] the optimal gain matrix $\mathbf{K}$ is

$$\mathbf{K} = \mathbf{P}\,\mathbf{H}'\,\mathbf{R}^{-1}$$

and the differential equation for the propagation of the covariance matrix of the error in the state vector $\mathbf{y}$ is

$$\dot{\mathbf{P}} = \mathbf{F}\,\mathbf{P} + \mathbf{P}\,\mathbf{F}' - \mathbf{P}\,\mathbf{H}'\,\mathbf{R}^{-1}\,\mathbf{H}\,\mathbf{P} + \mathbf{G}\,\mathbf{Q}\,\mathbf{G}' \quad (2)$$

The trajectory desired is that one which minimizes the total

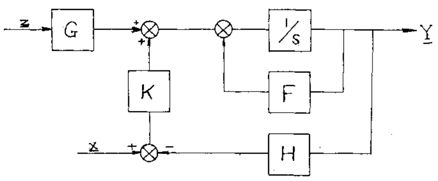

Fig. 1 Diagram of estimator

expected propellant consumption. As an approximation to this, the payoff quantity to be minimized was chosen as

$$\phi = (t_1 - t_0) + [a_1 P_{11} + a_2 P_{22} + a_3 P_{33}]_{t=t_1}$$

The first term represents the main phase propellant consumption, since b is nominally constant. The other three terms represent a weighted sum of the variances of each of the three state variables (y,v,u). By suitable choices of the weighting factors, ϕ can be made to vary, at least over a small range, in the same way as total propellant consumption.

Optimization Process

The process used in finding the optimal trajectory is the steepest ascent process.[3] The basic process will not be discussed. However, a modification in the operational technique was used in the study. The usual procedure[3] requires integration of the adjoint equations backward along a nominal, nonoptimum trajectory to evaluate the necessary change in the steering program. This requires storage of sufficient information at points spaced closely enough to permit accurate integration. As an alternative to storing this data, it was felt preferable to generate it as the adjoint equations are integrated. The known values of the adjoint variables are at the end point, which is why the adjoint equations usually are integrated backwards. In the present problem, however, it is not feasible to integrate the state equations backward, because the variance equation (2) is unstable in that direction. Instead, the transition matrix concept is used to anticipate the necessary initial values of the adjoint variables, thus permitting forward integration.

Consider a set of linear differential equations:

$$\dot{\boldsymbol{\lambda}} = -\mathbf{F}'\boldsymbol{\lambda} \quad (3)$$

If $\boldsymbol{\lambda}(t_0) = \mathbf{I}$, the identity, yields $\Lambda(t) = \boldsymbol{\Lambda}(t,t_0)$, then, because of the superposition principle of linear equations, $\boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ must yield $\boldsymbol{\lambda}(t) = \boldsymbol{\Lambda}(t,t_0)\boldsymbol{\lambda}_0$. This fundamental matrix $\boldsymbol{\Lambda}(t,t_0)$ is the transition matrix for the set of linear equations. In this case, the adjoint equations are of this form. Thus the transition matrix $\boldsymbol{\Lambda}(t_1,t_0)$ can be found by a forward integration, and, with a desired final value of $\boldsymbol{\lambda}$, the necessary initial value can be found from

$$\boldsymbol{\lambda}(t_0) = \boldsymbol{\Lambda}^{-1}(t_1,t_0)\boldsymbol{\lambda}(t_1) \quad (4)$$

In this procedure, then, the state equations (1) and (2) and the adjoint equations (3) are integrated forward, the former from the known initial conditions and the latter from an identity. With the final value of the transition matrix, the desired initial adjoint matrix is found from the state transition equation (4). Then the full set of equations is integrated forward again, this time with the new initial adjoint matrix. This second integration is equivalent to the backwards integration of the usual procedure, and the necessary information for improving the trajectory is derived from it.

Assumptions and Results

For this example, the vehicle was assumed to have the capability of measuring its altitude and both velocity components relative to the surface directly beneath it; hence the measurement inference matrix $\mathbf{H}$ is an identity. The errors in the measurements were assumed to have no cross-correlation; thus the inverse of the measurement covariance matrix is

$$\mathbf{R}^{-1} = \begin{bmatrix} 1/\sigma_y^2 & 0 & 0 \\ 0 & 1/\sigma_v^2 & 0 \\ 0 & 0 & 1/\sigma_u^2 \end{bmatrix}$$

This matrix is a measure of the *certainty* of the measurements. A zero in one of the diagonal elements is equivalent to infinite *uncertainty*, which is a simple way of eliminating the measurement of any of the variables. The noise in the control system also was assumed to have no cross-correlation. Only the diagonal elements, σ_b^2 and σ_θ^2, are nonzero.

A vehicle was chosen with the following dynamic characteristics: $b = 31.8$ ft-sec^{-2}, $\gamma = 0.0025$ sec^{-1}. For the following boundary conditions:

$$y_0 = 289{,}800 \text{ ft} \qquad y_1 = 0$$
$$v_0 = -2283 \text{ ft-sec}^{-1} \qquad v_1 = 0$$
$$u_0 = 8147 \text{ ft-sec}^{-1} \qquad u_1 = 0$$

and with the assumption of 5.3 ft-sec^{-2} as the lunar gravity, the optimum (deterministic) steering program yielding minimum propellant consumption is

$$\tan\theta = 0.5 + 0.001t$$

The burning time required is 200 sec. This was used as the initial nominal trajectory for the optimization.

The errors in the estimate of the initial state were assumed to be represented by $P_{11} = 10^4$ ft^2, $P_{22} = P_{33} = 100$ ft^2-sec^{-2}. These values are not very important because of the exponential nature of the variance equation (2). More important is the noise in the measuring system, which was assumed to have

$$\sigma_y^2 = 10^4 + 10^{-6}\, y^2 \text{ ft}^2$$
$$\sigma_v^2 = 100 + 10^{-4}\, v^2 \text{ ft}^2\text{-sec}^{-2}$$
$$\sigma_u^2 = 100 + 10^{-4}\, u^2 \text{ ft}^2\text{-sec}^{-2}$$

Table 1 Performance of trajectories

No.	Description	t_1-t_0	P_{11}	P_{22}	P_{33}	ϕ $a_1 = 0.167$	ϕ $a_1 = 10$
1	Nominal; all measurements	200.0	796.1	5.2	8.3	603.5	24768.3
2	Optimal for $a_1 = 0.167$; all measurements	200.46	787.2	5.0	8.6	601.5	...
3	Optimal for $a_1 = 10$; all measurements	200.69	785.0	5.0	8.7	...	24634.9
4	Nominal; measure y	200.0	1110	7.3	159.2	580.5	...
5	Optimal for $a_1 = 0.167$; measure y	201.67	1098	6.9	159.7	569.4	...

The fixed terms are a guess at the effect of the surface roughness directly below the vehicle. The noise in the control system was assumed to have $\sigma_b^2 = 0.64$ ft²-sec⁻⁴ and $\sigma_\theta^2 = 10^{-4}$.

The effect of the weights given to the uncertainty terms was found by assuming them to be in the arbitrary ratio 1:160:100 and scaling them up or down together. Some results with the full set of measurements are shown as trajectories 1, 2, and 3 in Table 1. Note the small changes in the variances. The steering programs are shown in Fig. 2.

The effects of simpler measuring equipment were found by assuming σ_u and σ_v to be infinite, leaving only the measurement of altitude. In addition, the weight given to the variance of horizontal velocity was set equal to zero, since, in the parallel field, with only y being measured, there is no way to infer the horizontal velocity from the measurements. The results of this are in Table 1 as trajectories 4 and 5, and the steering programs are shown in Fig. 2.

The values of a_i used in the foregoing are arbitrary. To talk in terms of overall optimization, one must find values of a_i which correctly represent the trade-off between uncertainty and propellant during the final descent phase. Consider

$$m = m(t_1 - t_0, P_{11})$$

from which

$$\Delta m/\dot{m} = \Delta t_1 + (1/\dot{m})(\partial m/\partial P_{11})\Delta P_{11}$$

This is the same as $\Delta\phi$ if one neglects P_{22} and P_{33} and sets

$$a_1 = (1/\dot{m})(\partial m/\partial P_{11})$$

Now, if the vehicle makes its final descent from $y = 3(P_{11})^{1/2}$ at velocity v_d,

$$\Delta m_d = \dot{m}_d\, t_d = \dot{m}_d[3(P_{11})^{1/2}/v_d]$$

Thus

$$\partial m/\partial P_{11} = 3\dot{m}_d/2v_d(P_{11})^{1/2}$$

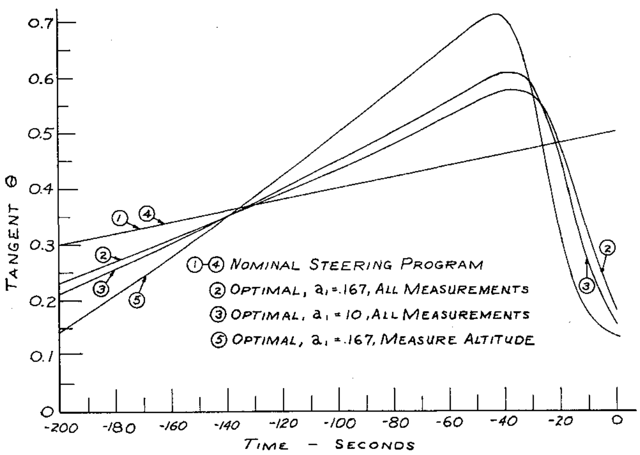

Fig. 2 Optimal steering programs

or

$$a_1 = (\dot{m}_d/\dot{m})[3/2v_d(P_{11})^{1/2}]$$

From the nominal trajectory, $P_{11} = 787$, $\dot{m}_d/\dot{m} = \frac{1}{12}$, and with $v_d = 10$ fps, $a_1 \cong 0.00045$.

Thus, for this example, unless the noise level is considerably larger than assumed here, the inclusion of the statistics is unimportant. However, the technique has been shown here to be feasible and is now available for more sensitive situations, such as atmospheric entry.

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Cross-Thermoelastic Phenomenon in Heterogeneous Aeolotropic Plates

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This note is concerned with the thermoelastic stress-strain relations in a heterogeneous aeolotropic plate theory that is based on the Euler-Bernoulli hypothesis.

Consider a thin elastic plate of constant thickness h which is heterogeneous in the thickness direction z. Let x, y be the

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